# The Energy-Momentum Tensor on $Spin^c$ Manifolds

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**Abstract**

On $\mathrm{Spin}^c$ manifolds, we study the Energy-Momentum tensor associated with a spinor field. First, we give a spinorial Gauss type formula for oriented hypersurfaces of a $\mathrm{Spin}^c$ manifold. Using the notion of generalized cylinders, we derive the variationnal formula for the Dirac operator under metric deformation and point out that the Energy-Momentum tensor appears naturally as the second fundamental form of an isometric immersion. Finally, we show that generalized $\mathrm{Spin}^c$ Killing spinors for Codazzi Energy-Momentum tensor are restrictions of parallel spinors.



**Keywords**: $\mathrm{Spin}^c$ structures; $\mathrm{Spin}^c$ Gauss formula; metric variation formula for the Dirac operator; Energy-Momentum tensor; generalized cylinder; generalized Killing spinors.


# 1 Introduction

In [14], O. Hijazi proved that on a compact Riemannian spin manifold $(M^n, g)$ any eigenvalue $\lambda$ of the Dirac operator to which is attached an eigenspinor $\psi$ satisfies

$$\lambda^2 \geqslant \inf_M \left(\frac{1}{4}\mathrm{Scal}^M + |\ell^\psi|^2\right), \tag{1}$$

where $\mathrm{Scal}^M$ is the scalar curvature of the manifold $M$ and $\ell^\psi$ is the field of symmetric endomorphisms associated with the field of quadratic forms $T^\psi$ called the Energy-Momentum tensor. It is defined on the complement set of zeroes of the eigenspinor $\psi$, for any vector $X \in \Gamma(TM)$ by

$$T^\psi(X) = \mathrm{Re} < X \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} > .$$

Here $\nabla$ denotes the Levi-Civita connection on the spinor bundle of $M$ and "$\cdot$" the Clifford multiplication. The limiting case of (1) is characterized by the existence of a spinor field $\psi$ satisfying for all $X \in \Gamma(TM)$,

$$\nabla_X \psi = -\ell^\psi(X) \cdot \psi. \tag{2}$$

For $\mathrm{Spin^c}$ structures, the complex line bundle $L^M$ is endowed with an arbitrary connection and hence an arbitrary curvature $i\Omega^M$ which is an imaginairy 2-form on the manifold. In terms of the Energy-Momentum tensor the author proved in [25] that on a compact Riemannian $\mathrm{Spin^c}$ manifold any eigenvalue $\lambda$ of the Dirac operator to which is attached an eigenspinor $\psi$ satisfies

$$\lambda^2 \geqslant \inf_M \Big(\frac{1}{4}\mathrm{Scal}^M - \frac{c_n}{4}|\Omega^M| + |\ell^\psi|^2\Big), \tag{3}$$

where $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$. The limiting case of (3) is characterized by the existence of a spinor field $\psi$ satisfying for every $X \in \Gamma(TM)$,

$$\begin{cases} \nabla_X^{\Sigma M} \psi = -\ell^\psi(X) \cdot \psi, \\ \\ \Omega^M \cdot \psi = i\frac{c_n}{2}|\Omega^M|\psi. \end{cases} \tag{4}$$

Here $\nabla^{\Sigma M}$ denotes the Levi-Civita connection on the $\mathrm{Spin^c}$ spinor bundle and "$\cdot$" the $\mathrm{Spin^c}$ Clifford multiplication. In [25], the author showed also that the sphere with a special $\mathrm{Spin^c}$ structure is a limiting manifold for (3).

Studying the Energy-Momentum tensor on a Riemannian or semi-Riemannian spin manifolds has been done by many authors, since it is related to several geometric constructions (see [12], [2], [24] and [6] for results in this topic). In this paper we study the Energy-Momentum tensor on Riemannian and semi-Riemannian $\mathrm{Spin^c}$ manifolds. First, we prove that the Energy-Momentum tensor appears in the study of the variations of the spectrum of the Dirac operator:

**Proposition 1.1** *Let $(M^n, g)$ be a $\mathrm{Spin^c}$ Riemannian manifold and $g_t = g + tk$ a smooth 1-parameter family of metrics. For any spinor field $\psi \in \Gamma(\Sigma M)$, we have*

$$\frac{d}{dt}\Big|_{t=0} (D^{M_t}\tau_0^t\psi, \tau_0^t\psi)_{g_t} = -\frac{1}{2}\int_M < k, T_\psi > dv_g, \tag{5}$$

*where* $(.,.) = \int_M \mathrm{Re}\,\langle .,.\rangle\, dv_g$, *the Dirac operator* $D^{M_t}$ *is the Dirac operator associated with* $M_t = (M, g_t)$ *and* $\tau_0^t\psi$ *is the image of* $\psi$ *under the isometry* $\tau_0^t$ *between the spinor bundles of* $(M, g)$ *and* $(M, g_t)$. *Here* $T_\psi$ *is defined by* $T_\psi = |\psi|^2 T^\psi$ *and* $T^\psi$ *is the symmetric bilinear form associated with the Energy-Momentum tensor, i.e. it is given for every* $X, Y \in \Gamma(TM)$ *by* $T^\psi(X,Y) = \frac{1}{2}\mathrm{Re}\left\langle X\cdot\nabla_Y\psi + Y\cdot\nabla_X\psi, \frac{\psi}{|\psi|^2}\right\rangle$.

This was proven in [4] by J. P. Bourguignon and P. Gauduchon for spin manifolds. Using this, we extend to $\mathrm{Spin^c}$ manifolds a result by Th. Friedrich and E. C. Kim in [8] on spin manifolds:

**Theorem 1.2** *Let* $M$ *be a* $\mathrm{Spin^c}$ *Riemannian manifold. A pair* $(g_0, \psi_0)$ *is a critical point of the Lagrange functional*

$$\mathcal{W}(g,\psi) = \int_U \Big(\mathrm{Scal}_g^M + \varepsilon\lambda|\psi|_g^2 - \varepsilon\mathrm{Re} < D_g\psi, \psi >_g \Big) dv_g,$$

$(\lambda, \varepsilon \in \mathbb{R})$ *for all open subsets* $U$ *of* $M$ *if and only if* $(g_0, \psi_0)$ *is a solution of the following system*

$$\begin{cases} D_g\psi = \lambda\psi, \\ \mathrm{ric}_g^M - \frac{\mathrm{Scal}_g^M}{2}\, g = \frac{\varepsilon}{2} T_\psi, \end{cases}$$

*where* $\mathrm{ric}_g^M$ *denotes the Ricci curvature of* $M$ *considered as a symmetric bilinear form.*

Now, we interprete the Energy-Momentum tensor as the second fundamental form of a hypersurface. In fact, we prove the following:

**Proposition 1.3** *Let* $M^n \hookrightarrow (\mathcal{Z}, g)$ *be any compact oriented hypersurface isometrically immersed in an oriented Riemannian* $\mathrm{Spin^c}$ *manifold* $(\mathcal{Z}, g)$, *of mean curvature* $H$ *and Weingarten map* $W$. *Assume that* $\mathcal{Z}$ *admits a parallel spinor field* $\psi$, *then the Energy-Momentum tensor associated with* $\varphi =: \psi_{|M}$ *satisfies*

$$2\ell^\varphi = -W.$$

*Moreover, if the mean curvature* $H$ *is constant, the hypersurface* $M$ *satisfies the equality case in (3) if and only if*

$$\mathrm{Scal}^{\mathcal{Z}} - 2\,\mathrm{ric}^{\mathcal{Z}}(\nu,\nu) - c_n|\Omega^M| = 0. \tag{6}$$

This was proven by Morel in [24] for a compact oriented hypersurface of a spin manifold carrying parallel spinor but in this case the hypersurface $M$ is directly a limiting manifold for (1) without the condition (6).

Finally, we study generalized Killing spinors on $\mathrm{Spin}^c$ manifolds. They are characterized by the identity, for any tangent vector field $X$ on $M$,

$$\nabla_X^{\Sigma M}\psi = \frac{1}{2}F(X)\cdot\psi, \tag{7}$$

where $F$ is a given symmetric endomorphism on the tangent bundle. It is straightforward to see that

$$2T^\psi(X,Y) = -\langle F(X), Y\rangle .$$

These spinors are closely related to the so-called $T$–Killing spinors studied by Friedrich and Kim in [9] on spin manifolds. It is natural to ask whether the tensor $F$ can be realized as the Weingarten tensor of some isometric embedding of $M$ in a manifold $\mathcal{Z}^{n+1}$ carrying parallel spinors. Morel studied this problem in the case of spin manifolds where the tensor $F$ is parallel and in [2], the authors studied the problem in the case of semi-Riemannian spin manifolds where the tensor $F$ is a Codazzi-Mainardi tensor. We establish the corresponding result for semi-Riemannian $\mathrm{Spin}^c$ manifolds:

**Theorem 1.4** *Let $(M^n, g)$ be a semi-Riemannian $\mathrm{Spin}^c$ manifold carrying a generalized $\mathrm{Spin}^c$ Killing spinor $\varphi$ with a Codazzi-Mainardi tensor $F$. Then the generalized cylinder $\mathcal{Z} := I\times M$ with the metric $dt^2 + g_t$, where $g_t(X,Y) = g((\mathrm{Id} - tF)^2X, Y)$, equipped with the $\mathrm{Spin}^c$ structure arising from the given one on $M$ has a parallel spinor whose restriction to $M$ is just $\varphi$.*

A characterisation of limiting 3-dimensional manifolds for (3), having generalized $\mathrm{Spin}^c$ Killing spinors with Codazzi tensor is then given.

The paper is organised as follows: In Section 2, we collect basic material on spinors and the Dirac operator on semi-Riemannian $\mathrm{Spin}^c$ manifolds. In Section 3, we study hypersurfaces of $\mathrm{Spin}^c$ manifolds. We derive a spinorial Gauss formula after identifying the restriction of the $\mathrm{Spin}^c$ spinor bundle of the ambient manifold with the $\mathrm{Spin}^c$ spinor bundle of the hypersurface. In Section 4, we define the generalized cylinder of a $\mathrm{Spin}^c$ manifold $M$ and we collect formulas relating the curvature of a generalized cylinder to geometric data on $M$. In section 5, we compare the Dirac operators for two differents semi-Riemannian metrics, then one first has to identify the spinor bundles using parallel transport. In the last section, we interprete the Energy-Momentum tensor as the second fundamental form of a hypersurface and we study generalized $\mathrm{Spin}^c$ Killing spinors. The author would like to thank Oussama Hijazi for his support and encouragements.

## 2 The Dirac operator on semi-Riemannian Spin$^c$ manifolds

In this section, we collect some algebraic and geometric preliminaries concerning the Dirac operator on semi-Riemannian $\mathrm{Spin}^c$ manifolds. Details can be found in [3] and [2]. Let $r + s = n$ and consider on $\mathbb{R}^n$ the nondegenerate symmetric bilinear form of signature $(r, s)$ given by

$$\langle v, w\rangle := \sum_{j=1}^{r} v_j w_j - \sum_{j=r+1}^{n} v_j w_j,$$

for any $v$, $w \in \mathbb{R}^n$. We denote by $\mathrm{Cl}_{r,s}$ the real Clifford algebra corresponding to $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, this is the unitary algebra generated by $\mathbb{R}^n$ subject to the relations

$$e_j \cdot e_k + e_k \cdot e_j = \begin{cases} -2\delta_{jk} & \text{if} \quad j \leqslant r, \\ \ \ 2\delta_{jk} & \text{if} \quad j > r, \end{cases}$$

where $(e_j)_{1\leqslant j\leqslant n}$ is an orthonormal basis of $\mathbb{R}^n$ of signature $(r, s)$, i.e., $\langle e_j, e_k\rangle = \varepsilon_j\delta_{jk}$ and $\varepsilon_j = \pm 1$. The complex Clifford algebra $\mathbb{C}\mathrm{l}_{r,s}$ is the complexification of $\mathrm{Cl}_{r,s}$ and it decomposes into even and odd elements $\mathbb{C}\mathrm{l}_{r,s} = \mathbb{C}\mathrm{l}^0_{r,s} \oplus \mathbb{C}\mathrm{l}^1_{r,s}$. The real spin group is defined by

$$\mathrm{Spin}(r, s) := \{v_1 \cdot ... \cdot v_{2k} \in \mathrm{Cl}_{r,s} \mid v_j \in \mathbb{R}^n \text{ such that } \langle v_j, v_j\rangle = \pm 1\}.$$

The spin group $\mathrm{Spin}(r, s)$ is the double cover of $\mathrm{SO}(r, s)$, in fact the following sequence is exact

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}(r, s) \xrightarrow{\xi} \mathrm{SO}(r, s) \longrightarrow 1,$$

where $\xi = \mathrm{Ad}_{|\mathrm{Spin}(r,s)}$ and $\mathrm{Ad}$ is defined by

$$\begin{array}{rcl} \mathrm{Ad} : \mathrm{Cl}^*_{r,s} & \longrightarrow & \mathrm{End}(\mathbb{R}^n) \\ w & \longrightarrow & \mathrm{Ad}_w : v \longrightarrow \mathrm{Ad}_w(v) = w \cdot v \cdot w^{-1}. \end{array}$$

Here $\mathrm{Cl}^*_{r,s}$ denotes the group of units of $\mathrm{Cl}_{r,s}$. Since $\mathbb{S}^1 \cap \mathrm{Spin}(r, s) = \{\pm 1\}$, we define the complex spin group by

$$\mathrm{Spin}^c(r, s) = \mathrm{Spin}(r, s) \times_{\mathbb{Z}_2} \mathbb{S}^1.$$

The complex spin group is the double cover of $\mathrm{SO}(r, s) \times \mathbb{S}^1$, this yields to the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(r, s) \xrightarrow{\xi^c} \mathrm{SO}(r, s) \times \mathbb{S}^1 \longrightarrow 1,$$

where $\xi^c = (\xi, \mathrm{Id}^2)$. When $n = 2m$ is even, $\mathbb{C}\mathrm{l}_{r,s}$ has a unique irreducible complex representation $\chi_{2m}$ of complex dimension $2^m$, $\chi_{2m} : \mathbb{C}\mathrm{l}_{r,s} \longrightarrow \mathrm{End}(\Sigma_{r,s})$. If $n =$

$2m+1$ is odd, $\mathbb{C}\mathrm{l}_{r,s}$ has two inequivalent irreducible representations both of complex dimension $2^m$, $\chi^j_{2m+1} : \mathbb{C}\mathrm{l}_{r,s} \longrightarrow \mathrm{End}(\Sigma^j_{r,s})$, for $j = 0$ or $1$, where $\Sigma^j_{r,s} = \{\sigma \in \Sigma_{r,s},\ \chi^j_{2m+1}(\omega_{r,s})\sigma = (-1)^j\sigma\}$ and $\omega_{r,s}$ is the complex volume element

$$\omega_{r,s} = \begin{cases} i^{m-s} & e_1 \cdot \ldots \cdot e_n \qquad \text{if } n = 2m, \\ i^{m-1+s} & e_1 \cdot \ldots \cdot e_n \qquad \text{if } n = 2m+1. \end{cases}$$

We define the complex spinorial representation $\rho_n$ by the restriction of an irreducible representation of $\mathbb{C}\mathrm{l}_{r,s}$ to $\mathrm{Spin}^c(r,s)$:

$$\rho_n := \begin{cases} \chi_{2m|_{\mathrm{Spin}^c(r,s)}} & \text{if } n = 2m, \\ \\ \chi^0_{2m+1|_{\mathrm{Spin}^c(r,s)}} & \text{if } n = 2m+1, \end{cases}.$$

When $n = 2m$ is even, $\rho_n$ decomposes into two inequivalent irreductible representations $\rho_n^+$ and $\rho_n^-$, i.e., $\rho_n = \rho_n^+ + \rho_n^- : \mathrm{Spin}^c(r,s) \to \mathrm{Aut}(\Sigma_{r,s})$. The space $\Sigma_{r,s}$ decomposes into $\Sigma_{r,s} = \Sigma^+_{r,s} \oplus \Sigma^-_{r,s}$, where $\omega_{r,s}$ acts on $\Sigma^+_{r,s}$ as the identity and minus the identity on $\Sigma^-_{r,s}$. If $n = r+s$ is odd and when restricted to $\mathrm{Spin}^c(r,s)$, the two representations $\chi^0_{2m+1|_{\mathrm{Spin}^c(r,s)}}$ and $\chi^1_{2m+1|_{\mathrm{Spin}^c(r,s)}}$ are equivalent and we simply choose $\Sigma_{r,s} := \Sigma^0_{r,s}$. The complex spinor bundle $\Sigma_{r,s}$ carries a Hemitian symmetric bilinear $\mathrm{Spin}^c(r,s)$-invariant form $\langle \cdot, \cdot \rangle$, such that

$$\langle v \cdot \sigma_1, \sigma_2 \rangle = (-1)^{s+1} \langle \sigma_1, v \cdot \sigma_2 \rangle \text{ for all } \sigma_1, \sigma_2 \in \Sigma_{r,s} \text{ and } v \in \mathbb{R}^n.$$

Now, we give the following isomorphism $\alpha$, which is of particular importance for the identification of the $\mathrm{Spin^c}$ bundles in the context of immersions of hypersurfaces:

$$\begin{aligned} \alpha : \mathbb{C}\mathrm{l}_{r,s} &\to \mathbb{C}\mathrm{l}^0_{r+1,s} \\ e_j &\to \nu \cdot e_j, \end{aligned} \tag{8}$$

where we look at an embedding of $\mathbb{R}^n$ onto $\mathbb{R}^{n+1}$ such that $(\mathbb{R}^n)^\perp$ is spacelike and spanned by a spacelike unit vector $\nu$.

Let $N^n$ be an oriented semi-Riemannian manifold of signature $(r,s)$ and let $P_{\mathrm{SO}}N$ be the $\mathrm{SO}(r,s)$-principal bundle of positively space and time oriented orthonormal tangent frames. A complex $\mathrm{Spin^c}$ structure on $N$ is a $\mathrm{Spin}^c(r,s)$-principal bundle $P_{\mathrm{Spin^c}}N$ over $N$, an $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1}N$ over $N$ together with a twofold covering map $\Theta : P_{\mathrm{Spin^c}}N \longrightarrow P_{\mathrm{SO}}N \times_N P_{\mathbb{S}^1}N$ such that

$$\Theta(ua) = \Theta(u)\xi^c(a),$$

for every $u \in P_{\mathrm{Spin^c}}N$ and $a \in \mathrm{Spin}^c(r,s)$, i.e., $N$ has a $\mathrm{Spin^c}$ structure if and only if there exists an $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1}N$ over $N$ such that the transition functions $g_{\alpha\beta} \times l_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{SO}(r,s) \times \mathbb{S}^1$ of the

$\mathrm{SO}(r,s)\times\mathbb{S}^1$-principal bundle $P_{\mathrm{SO}}N\times_N P_{\mathbb{S}^1}N$ admit lifts to $\mathrm{Spin}^c(r,s)$ denoted by $\widetilde{g}_{\alpha\beta}\times\widetilde{l}_{\alpha\beta}: U_\alpha\cap U_\beta\longrightarrow\mathrm{Spin}^c(r,s)$, such that $\xi^c\circ(\widetilde{g}_{\alpha\beta}\times\widetilde{l}_{\alpha\beta}) = g_{\alpha\beta}\times l_{\alpha\beta}$. This, anyhow, is equivalent to the second Stiefel-Whitney class $w_2(N)$ being equal, modulo 2, to the first chern class $c_1(L^N)$ of the complex line bundle $L^N$. It is the complex line bundle associated with the $\mathbb{S}^1$-principal fibre bundle via the standard representation of the unit circle.

Let $\Sigma N := P_{\mathrm{Spin}^c}N\times_{\rho_n}\Sigma_{r,s}$ be the spinor bundle associated with the spinor representation. A section of $\Sigma N$ will be called a spinor field. Using the cocycle condition of the transition functions of the two principal fibre bundles $P_{\mathrm{Spin}^c}N$ and $P_{\mathrm{SO}}N\times_N P_{\mathbb{S}^1}N$, we can prove that

$$\Sigma N = \Sigma' N\otimes (L^N)^{\frac{1}{2}},$$

where $\Sigma' N$ is the locally defined spin bundle and $(L^N)^{\frac{1}{2}}$ is locally defined too but $\Sigma N$ is globally defined. The tangent bundle $TN = P_{\mathrm{SO}}N\times_{\rho_0}\mathbb{R}^n$ where $\rho_0$ stands for the standard matrix representation of $\mathrm{SO}(r,s)$ on $\mathbb{R}^n$, can be seen as the associated vector bundle $TN\simeq P_{\mathrm{Spin}^c}N\times_{pr_1\circ\xi^c\circ\rho_0}\mathbb{R}^n$ where $pr_1$ is the first projection. One defines the Clifford multiplication at every point $p\in N$:

$$\begin{array}{rcl} T_pN\otimes\Sigma_pN & \longrightarrow & \Sigma_pN \\ {[b,v]}\otimes[b,\sigma] & \longrightarrow & [b,v]\cdot[b,\sigma] := [b, v\cdot\sigma = \chi_n(v)\sigma], \end{array}$$

where $b\in P_{\mathrm{Spin}^c}N$, $v\in\mathbb{R}^n$, $\sigma\in\Sigma_{r,s}$ and $\chi_n=\chi_{2m}$ if $n$ is even and $\chi_n=\chi^0_{2m+1}$ if $n$ is odd. The Clifford multiplication can be extended to differential forms. Clifford multiplication inherits the relations of the Clifford algebra, i.e., for $X,Y\in T_pN$ and $\varphi\in\Sigma_pN$ we have $X\cdot Y\cdot\varphi + Y\cdot X\cdot\varphi = -2\langle X,Y\rangle\varphi$. In even dimensions the spinor bundle splits into $\Sigma N = \Sigma^+N\oplus\Sigma^-N$, where $\Sigma^\pm N = P_{\mathrm{Spin}^c}N\times_{\rho_n^\pm}\Sigma^\pm_{r,s}$. Clifford multiplication by a non-vanishing tangent vector interchanges $\Sigma^+N$ and $\Sigma^-N$. The $\mathrm{Spin}^c(r,s)$-invariant nondegenerate symmetric sesquilinear form on $\Sigma_{r,s}$ and $\Sigma^\pm_{r,s}$ induces inner products on $\Sigma N$ and $\Sigma^\pm N$ which we again denote by $\langle\cdot,\cdot\rangle$ and it satisfies

$$\langle X\cdot\psi,\varphi\rangle = (-1)^{s+1}\langle\psi, X\cdot\varphi\rangle,$$

for every $X\in\Gamma(TN)$ and $\psi,\varphi\in\Gamma(\Sigma N)$. Additionally, given a connection 1-form $A^N$ on $P_{\mathbb{S}^1}N$, $A^N: T(P_{\mathbb{S}^1}N)\longrightarrow i\mathbb{R}$ and the connection 1-form $\omega^N$ on $P_{\mathrm{SO}}N$ for the Levi-Civita connection $\nabla^N$, we can define the connection

$$\omega^N\times A^N: T(P_{\mathrm{SO}}N\times_N P_{\mathbb{S}^1}N)\longrightarrow\mathfrak{so}_n\oplus i\mathbb{R} = \mathfrak{spin}^{\mathbb{C}}_n$$

on the principal fibre bundle $P_{\mathrm{SO}}N\times_N P_{\mathbb{S}^1}N$ and hence a covariant derivative $\nabla^{\Sigma N}$ on $\Sigma N$ [7] given locally by

$$\begin{aligned} \nabla^{\Sigma N}_{e_k}\varphi &= \Big[\widetilde{b\times s}, e_k(\sigma) + \frac{1}{4}\sum_{j=1}^n\varepsilon_j\,e_j\cdot\nabla^N_{e_k}e_j\cdot\sigma + \frac{1}{2}A^N(s_*(e_k))\sigma\Big] \\ &= e_k(\varphi) + \frac{1}{4}\sum_{j=1}^n\varepsilon_j\,e_j\cdot\nabla^N_{e_k}e_j\cdot\varphi + \frac{1}{2}A^N(s_*(e_k))\varphi, \end{aligned} \tag{9}$$

where $\varphi = [\widetilde{b \times s}, \sigma]$ is a locally defined spinor field, $b = (e_1, \dots, e_n)$ is a local space and time oriented orthonormal tangent frame, $s : U \longrightarrow P_{\mathbb{S}^1}N$ is a local section of $P_{\mathbb{S}^1}N$ and $\widetilde{b \times s}$ is the lift of the local section $b \times s : U \to P_{\mathrm{SO}}N \times_N P_{\mathbb{S}^1}N$ to the 2-fold covering $\Theta : P_{\mathrm{Spin}^c}N \longrightarrow P_{\mathrm{SO}}N \times_N P_{\mathbb{S}^1}N$. The curvature of $A^N$ is an imaginary valued 2-form denoted by $F_{A^N} = dA^N$, i.e., $F_{A^N} = i\Omega^N$, where $\Omega^N$ is a real valued 2-form on $P_{\mathbb{S}^1}N$. We know that $\Omega^N$ can be viewed as a real valued 2-form on $N$ [7]. In this case $i\Omega^N$ is the curvature form of the associated line bundle $L^N$. The curvature tensor $\mathcal{R}^{\Sigma N}$ of $\nabla^{\Sigma N}$ is given by

$$\mathcal{R}^{\Sigma N}(X,Y)\varphi = \frac{1}{4}\sum_{j,k=1}^{n} \varepsilon_j\varepsilon_k \left\langle R^N(X,Y)e_j, e_k\right\rangle e_j \cdot e_k \cdot \varphi + \frac{i}{2}\Omega^N(X,Y)\varphi, \tag{10}$$

where $R^N$ is the curvature tensor of the Levi-Civita connection $\nabla^N$. In the $\mathrm{Spin}^c$ case, the Ricci identity translates, for every $X \in \Gamma(TN)$, to

$$\sum_{k=1}^{n} \varepsilon_k\, e_k \cdot \mathcal{R}^{\Sigma N}(e_k, X)\varphi = \frac{1}{2}\mathrm{Ric}^N(X) \cdot \varphi - \frac{i}{2}(X \lrcorner \Omega^N) \cdot \varphi. \tag{11}$$

Here $\mathrm{Ric}^N$ denotes the Ricci curvature considered as a field of endomorphism on $TN$. The Ricci curvature considered as a symmetric bilinear form will be written $\mathrm{ric}^N(Y,Z) = \left\langle \mathrm{Ric}^N(Y), Z\right\rangle$. The Dirac operator maps spinor fields to spinor fields and is locally defined by

$$D^N\varphi = i^s \sum_{j=1}^{n} \varepsilon_j e_j \cdot \nabla^{\Sigma N}_{e_j}\varphi,$$

for every spinor field $\varphi$. The Dirac operator is an elliptic operator, formally selfadjoint, i.e. if $\psi$ or $\varphi$ has compact support, then $\int_N \left\langle D^N\varphi, \psi\right\rangle dv_g = \int_N \left\langle \varphi, D^N\psi\right\rangle dv_g$.

# 3 Semi-Riemannian Spin$^c$ hypersurfaces and the Gauss formula

In this section, we study $\mathrm{Spin}^c$ structures of hypersurfaces, such as the restriction of a $\mathrm{Spin}^c$ bundle of an ambient semi-Riemannian manifold and the complex spinorial Gauss formula.

Let $\mathcal{Z}$ be an oriented $(n+1)$-dimensional semi-Riemannian $\mathrm{Spin}^c$ manifold and $M \subset \mathcal{Z}$ a semi-Riemannian hypersurface with trivial spacelike normal bundle. This means that there is a vector field $\nu$ on $\mathcal{Z}$ along $M$ satisfying $\langle \nu, \nu\rangle = +1$ and $\langle \nu, TM\rangle = 0$. Hence if the signature of $M$ is $(r,s)$, then the signature of $\mathcal{Z}$ is $(r+1, s)$.

**Proposition 3.1** *The hypersurface $M$ inherts a* $\mathrm{Spin^c}$ *structure from that on $\mathcal{Z}$, and we have*

$$\begin{cases} \Sigma\mathcal{Z}_{|_M} \simeq \Sigma M & \text{if } n \text{ is even,} \\ \Sigma^+\mathcal{Z}_{|_M} \simeq \Sigma M & \text{if } n \text{ is odd.} \end{cases}$$

*Moreover Clifford multiplication by a vector field $X$, tangent to $M$, is given by*

$$X \bullet \varphi = (\nu \cdot X \cdot \psi)_{|_M}, \tag{12}$$

*where $\psi \in \Gamma(\Sigma\mathcal{Z})$ (or $\psi \in \Gamma(\Sigma^+\mathcal{Z})$ if $n$ is odd), $\varphi$ is the restriction of $\psi$ to $M$, "$\cdot$" is the Clifford multiplication on $\mathcal{Z}$, and "$\bullet$" that on $M$.*

**Proof:** The bundle of space and time oriented orthonormal frames of $M$ can be embedded into the bundle of space and time oriented orthonormal frames of $\mathcal{Z}$ restricted to $M$, by

$$\begin{array}{rccc} \Phi : & P_{\mathrm{SO}}M & \longrightarrow & P_{\mathrm{SO}}\mathcal{Z}_{|_M} \\ & (e_1, \cdots, e_n) & \longrightarrow & (\nu, e_1, \cdots, e_n). \end{array} \tag{13}$$

The isomorphism $\alpha$, defined in (8) yields the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spin}^c(r,s) & \hookrightarrow & \mathrm{Spin}^c(r+1,s) \\ \Big\downarrow \xi^c & & \xi^c \Big\downarrow \\ \mathrm{SO}(r,s) \times \mathbb{S}^1 & \hookrightarrow & \mathrm{SO}(r+1,s) \times \mathbb{S}^1 \end{array}$$

where the inclusion of $\mathrm{SO}(r,s)$ in $\mathrm{SO}(r+1,s)$ is that which fixes the first basis vector under the action of $\mathrm{SO}(r+1,s)$ on $\mathbb{R}^{n+1}$. This allows to pull back via $\Phi$ the principal bundle $P_{\mathrm{Spin^c}}\mathcal{Z}_{|_M}$ as a $\mathrm{Spin^c}$ structure for $M$, denoted by $P_{\mathrm{Spin^c}}M$. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} P_{\mathrm{Spin^c}}M & \longrightarrow & P_{\mathrm{Spin^c}}\mathcal{Z}_{|_M} \\ \Big\downarrow \Theta & & \Theta \Big\downarrow \\ P_{\mathrm{SO}}M \times_M P_{\mathbb{S}^1}\mathcal{Z}_{|_M} & \longrightarrow & P_{\mathrm{SO}}\mathcal{Z}_{|_M} \times_M P_{\mathbb{S}^1}\mathcal{Z}_{|_M} \end{array}$$

The $\mathrm{Spin}^c(r,s)$-principal bundle $(P_{\mathrm{Spin^c}}M, \pi, M)$ and the $\mathbb{S}^1$-principal bundle $(P_{\mathbb{S}^1}M =: P_{\mathbb{S}^1}\mathcal{Z}|_M, \pi, M)$ define a $\mathrm{Spin^c}$ structure on $M$. Let $\Sigma\mathcal{Z}$ be the spinor bundle on $\mathcal{Z}$,

$$\Sigma\mathcal{Z} = P_{\mathrm{Spin^c}}\mathcal{Z} \times_{\rho_{n+1}} \Sigma_{r+1,s},$$

where $\rho_{n+1}$ stands for the spinorial representation of $\mathrm{Spin}^c(r+1,s)$. Moreover, for any spinor $\psi = [\widetilde{b \times s}, \sigma] \in \Sigma\mathcal{Z}$ we can always assume that $pr_1 \circ \Theta(\widetilde{b \times s}) = b$ is a local section of $P_{\mathrm{SO}}\mathcal{Z}$ with $\nu$ for first basis vector where $pr_1$ is the projection into $P_{\mathrm{SO}}\mathcal{Z}$ . Then we have

$$\psi_{|_M} = [\widetilde{b \times s}_{|_{U\cap M}}, \sigma_{|_{U\cap M}}],$$

where the equivalence class is reduced to elements of $\mathrm{Spin}^c(r,s)$. It follows that one can realise the restriction to $M$ of the spinor bundle $\Sigma\mathcal{Z}$ as

$$\Sigma\mathcal{Z}_{|_M} = P_{\mathrm{Spin^c}}M \times_{\rho_{n+1}\circ\alpha} \Sigma_{r+1,s}.$$

If $n = 2m$ is even, it is easy to check that $\chi^0_{2m+1}\circ\alpha = \chi^0_{2m+1}{}_{|_{\mathbb{C}l^0_{r+1,s}}}$. Hence $\chi^0_{2m+1}\circ\alpha$ is an irreductible representation of $\mathbb{C}l_{r,s}$ of dimension $2^m$, as $\chi^0_{2m+1}{}_{|_{\mathbb{C}l^0_{r+1,s}}}$, and finally $\chi^0_{2m+1}\circ\alpha \cong \chi_{2m}$. We conclude that

$$\rho_{2m+1}\circ\alpha \cong \rho_{2m}, \quad \text{and} \quad \Sigma\mathcal{Z}_{|_M} \cong \Sigma M.$$

If $n = 2m+1$ is odd, we know that $\chi^0_{2m+1}$ is the unique irreductible representation of $\mathbb{C}l_{r,s}$ of dimension $2^m$ for which the action of the complex volume form is the identity. Since $n+1 = 2m+2$ is even, $\Sigma\mathcal{Z}$ decomposes into positive and negative parts, $\Sigma^{\pm}\mathcal{Z} = P_{\mathrm{Spin^c}}\mathcal{Z}\times_{\rho^{\pm}_{2m+1}} \Sigma^{\pm}_{r+1,s}$. It is easy to show that $\chi_{2m+2}\circ\alpha = \chi_{2m+2}{}_{|_{\mathbb{C}l^0_{r+1,s}}}$, but $\chi_{2m+2}\circ\alpha$ can be written as the direct sum of two irreductible inequivalent representations, as $\chi_{2m+2}{}_{|_{\mathbb{C}l^0_{r+1,s}}}$. Hence, we have

$$\chi_{2m+2}\circ\alpha = (\chi_{2m+2}\circ\alpha)^+ \oplus (\chi_{2m+1}\circ\alpha)^-,$$

where $(\chi_{2m+2}\circ\alpha)^{\pm}(\omega_{r,s}) = \pm\mathrm{Id}_{\Sigma_{r,s}}$. The representation $\chi^0_{2m+1}$ being the unique representation of $\mathbb{C}l_{r,s}$ of dimension $2^m$ for which the action of the volume form is the identity, we get $(\chi_{2m+2}\circ\alpha)^+ \cong \chi^0_{2m+1}$. Finally,

$$\rho^+_{2m+2}\circ\alpha \cong \rho_{2m+1} \quad \text{and} \quad \Sigma^+\mathcal{Z}_{|_M} \cong \Sigma M.$$

Now, Equation (12) follows directly from the above identification.

**Remarks 3.2** 1. *The algebraic remarks in the previous section show that if $n$ is odd we can also get $\Sigma^-\mathcal{Z}_{|_M} \simeq \Sigma M$, where the Clifford multiplication by a vector field tangent to $M$ is given by $X \bullet \varphi = -(\nu\cdot X\cdot\psi)_{|_M}$.*
2. *The connection 1-form defined on the restricted $\mathbb{S}^1$-principal bundle $(P_{\mathbb{S}^1}M =: P_{\mathbb{S}^1}\mathcal{Z}_{|_M}, \pi, M)$, is given by*

$$A^M = A^{\mathcal{Z}}{}_{|_M} : T(P_{\mathbb{S}^1}M) = T(P_{\mathbb{S}^1}\mathcal{Z})_{|_M} \longrightarrow i\mathbb{R}.$$

*Then the curvature 2-form $i\Omega^M$ on the $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1}M$ is given by $i\Omega^M = i\Omega^{\mathcal{Z}}{}_{|_M}$, which can be viewed as an imaginary 2-form on $M$ and hence as the curvature form of the line bundle $L^M$, the restriction of the line bundle $L^{\mathcal{Z}}$ to $M$.*
3. *For every $\psi\in\Gamma(\Sigma\mathcal{Z})$ ($\psi\in\Gamma(\Sigma^+\mathcal{Z})$ if $n$ is odd), the real 2-forms $\Omega^M$ and $\Omega^{\mathcal{Z}}$ are related by the following formulas:*

$$|\Omega^{\mathcal{Z}}|^2 = |\Omega^M|^2 + |\nu\lrcorner\Omega^{\mathcal{Z}}|^2, \tag{14}$$

$$(\Omega^{\mathcal{Z}} \cdot \psi)_{|_M} = \Omega^M \bullet \varphi + (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi. \tag{15}$$

*In fact, we can write*

$$\Omega^{\mathcal{Z}} = \sum_{i=1}^{n} \Omega^{\mathcal{Z}}(\nu, e_i)\, \nu \wedge e_i + \sum_{i<j}^{n} \Omega^{\mathcal{Z}}\ (e_i, e_j)\, e_i \wedge e_j = -(\nu \lrcorner \Omega^{\mathcal{Z}}) \wedge \nu + \Omega^M,$$

*which is (14). When restricting the Clifford multiplication of $\Omega^{\mathcal{Z}}$ by $\psi$ to the hypersurface $M$ we obtain*

$$(\Omega^{\mathcal{Z}} \cdot \psi)_{|_M} = (\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi)_{|_M} + (\Omega^M \cdot \psi)_{|_M} = (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi + \Omega^M \bullet \varphi. \tag{16}$$

**Proposition 3.3 (The spinorial Gauss formula)** *We denote by $\nabla^{\Sigma\mathcal{Z}}$ the spinorial Levi-Civita connection on $\Sigma\mathcal{Z}$ and by $\nabla^{\Sigma M}$ that on $\Sigma M$. For all $X \in \Gamma(TM)$ and for every spinor field $\psi \in \Gamma(\Sigma\mathcal{Z})$, then*

$$(\nabla^{\Sigma\mathcal{Z}}_X \psi)_{|_M} = \nabla^{\Sigma M}_X \varphi - \frac{1}{2} W(X) \bullet \varphi, \tag{17}$$

*where $W$ denotes the Weingarten map with respect to $\nu$ and $\varphi = \psi_{|_M}$. Moreover, let $D^{\mathcal{Z}}$ and $D^M$ be the Dirac operators on $\mathcal{Z}$ and $M$. Denoting by the same symbol any spinor and it's restriction to $M$, we have*

$$\nu \cdot D^{\mathcal{Z}} \varphi = \widetilde{D} \varphi + \frac{i^s n}{2} H \varphi - i^s \nabla^{\Sigma\mathcal{Z}}_\nu \varphi, \tag{18}$$

*where $H = \frac{1}{n}\mathrm{tr}(W)$ denotes the mean curvature and $\widetilde{D} = D^M$ if $n$ is even and $\widetilde{D} = D^M \oplus (-D^M)$ if $n$ is odd.*

**Proof:** The Riemannian Gauss formula is given, for every vector fields $X$ and $Y$ on $M$, by

$$\nabla^{\mathcal{Z}}_X Y = \nabla^M_X Y + \langle W(X), Y \rangle\, \nu. \tag{19}$$

Let $(e_1, e_2, ..., e_n)$ a local space and time oriented orthonormal frame of $M$, such that $b = (e_0 = \nu, e_1, e_2, ..., e_n)$ is that of $\mathcal{Z}$. We consider $\psi$ a local section of $\Sigma\mathcal{Z}$, $\psi = [\widetilde{b \times s}, \sigma]$ where $s$ is a local section of $P_{\mathbb{S}^1}\mathcal{Z}$. Using (9), (19) and the fact that $X(\psi)|_M = X(\varphi)$ for $X \in \Gamma(TM)$, we compute for $j = 1, ..., n$

$$\begin{aligned}
\left(\nabla^{\Sigma\mathcal{Z}}_{e_j} \psi\right)_{|_M} &= e_j(\varphi) + \frac{1}{4} \sum_{k=0}^{n} \varepsilon_k (e_k \cdot \nabla^{\mathcal{Z}}_{e_j} e_k \cdot \psi)_{|_M} + \frac{1}{2} A^Z(s_*(e_j))\varphi \\
&= e_j(\varphi) + \frac{1}{4} \sum_{k=1}^{n} \varepsilon_k (e_k \cdot \nabla^{\mathcal{Z}}_{e_j} e_k \cdot \psi)_{|_M} + \frac{1}{4} (\nu \cdot \nabla^{\mathcal{Z}}_{e_j} \nu \cdot \psi)_{|_M} + \frac{1}{2} A^M(s_*(e_j))\varphi \\
&= \nabla^{\Sigma M}_{e_j} \varphi + \frac{1}{4} \sum_{k=1}^{n} \varepsilon_k < W(e_j), e_k > (e_k \cdot \nu \cdot \psi)_{|_M} - \frac{1}{4} (\nu \cdot W(e_j) \cdot \psi)_{|_M} \\
&= \nabla^{\Sigma M}_{e_j} \varphi - \frac{1}{2} (\nu \cdot W(e_j) \cdot \psi)_{|_M} \\
&= \nabla^{\Sigma M}_{e_j} \varphi - \frac{1}{2} W(e_j) \bullet \varphi.
\end{aligned}$$

Moreover $(D^{\mathcal{Z}}\psi)_{|M} = i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla^{\Sigma\mathcal{Z}}_{e_j}\psi)_{|M} + i^s(\nu \cdot \nabla^{\Sigma\mathcal{Z}}_{\nu}\psi)_{|M}$, and by (17),

$$\begin{aligned} i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla^{\Sigma\mathcal{Z}}_{e_j}\psi)_{|M} &= i^s \sum_{j=1}^n \varepsilon_j \,(e_j \cdot \nabla^{\Sigma M}_{e_j}\varphi) - i^s \frac{1}{2} \sum_{j=1}^n \varepsilon_j \,(e_j \cdot \nu \cdot W(e_j) \cdot \psi)_{|M} \\ &= -i^s \nu \cdot \sum_{j=1}^n \varepsilon_j \, \nu \cdot e_j \cdot \nabla^{\Sigma M}_{e_j}\varphi + i^s \frac{1}{2} \sum_{j=1}^n \varepsilon_j \,(\nu \cdot e_j \cdot W(e_j) \cdot \psi)_{|M} \\ &= -\nu \cdot \widetilde{D}\varphi - \frac{i^s}{2} \mathrm{tr}(W)(\nu \cdot \psi)_{|M}. \end{aligned}$$

**Proposition 3.4** *Let $\mathcal{Z}$ be an $(n+1)$-dimensional semi-Riemannian* $\mathrm{Spin}^{\mathrm{c}}$ *manifold. Assume that $\mathcal{Z}$ carries a semi-Riemannian foliation by hypersurfaces with trivial spacelike normal bundle, i.e., the leaves $M$ are semi-Riemannian hypersurfaces and there exists a vector field $\nu$ on $\mathcal{Z}$ perpendicular to the leaves such that $\langle \nu, \nu \rangle = 1$ and $\nabla^{\mathcal{Z}}_{\nu}\nu = 0$. Then the commutator of the leafwise Dirac operator and the normal derivative is given by*

$$i^{-s}[\nabla^{\Sigma\mathcal{Z}}_{\nu}, \widetilde{D}]\,\varphi = \mathfrak{D}^W\varphi - \frac{n}{2}\,\nu \cdot \mathrm{grad}^M(H) \cdot \varphi + \frac{1}{2}\,\nu \cdot \mathrm{div}^M(W) \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi.$$

*Here* $\mathrm{grad}^M$ *denotes the leafwise gradient,* $\mathrm{div}^M(W) = \sum_{i=1}^n \varepsilon_i \,(\nabla^M_{e_i} W)(e_i)$ *denotes the leafwise divergence of the endomorphism field $W$ and $\mathfrak{D}^W\varphi = \sum_{i=1}^n \varepsilon_i \, \nu \cdot e_i \cdot \nabla^{\Sigma M}_{W(e_i)}\varphi$.*

**Proof:** We choose a local oriented orthonormal tangent frame $(e_1, \ldots, e_n)$ for the leaves and we may assume for simplicity that $\nabla^{\mathcal{Z}}_{\nu} e_j = 0$. Now, we compute

$$\begin{aligned} i^{-s}[\nabla^{\Sigma\mathcal{Z}}_{\nu}, \widetilde{D}]\,\varphi &= \sum_{j=1}^n \varepsilon_j \left( \nabla^{\Sigma\mathcal{Z}}_{\nu}(\nu \cdot e_j \cdot \nabla^{\Sigma M}_{e_j}\varphi) - \nu \cdot e_j \cdot \nabla^{\Sigma M}_{e_j} \nabla^{\Sigma\mathcal{Z}}_{\nu}\varphi \right) \\ &= \sum_{j=1}^n \varepsilon_j \, \nu \cdot e_j \cdot \left( \nabla^{\Sigma\mathcal{Z}}_{\nu} \nabla^{\Sigma M}_{e_j}\varphi - \nabla^{\Sigma M}_{e_j} \nabla^{\Sigma\mathcal{Z}}_{\nu}\varphi \right) \\ &\overset{(17)}{=} \sum_{j=1}^n \varepsilon_j \, \nu \cdot e_j \cdot \Big[ \nabla^{\Sigma\mathcal{Z}}_{\nu}(\nabla^{\Sigma\mathcal{Z}}_{e_j} + \frac{1}{2}\nu \cdot W(e_j)) \\ &\qquad -(\nabla^{\Sigma\mathcal{Z}}_{e_j} + \frac{1}{2}\nu \cdot W(e_j))\nabla^{\Sigma\mathcal{Z}}_{\nu} \Big]\varphi \\ &= \sum_{j=1}^n \varepsilon_j \, \nu \cdot e_j \cdot \left( \mathcal{R}^{\Sigma\mathcal{Z}}(\nu, e_j) + \nabla^{\Sigma\mathcal{Z}}_{[\nu, e_j]} + \frac{1}{2}\nu \cdot (\nabla^{\mathcal{Z}}_{\nu} W)(e_j) \right)\varphi \\ &\overset{(11)}{=} -\frac{1}{2}\nu \cdot \mathrm{Ric}^{\mathcal{Z}}(\nu) \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi \\ &\qquad + \sum_{j=1}^n \varepsilon_j \, \nu \cdot e_j \cdot \left( \nabla^{\Sigma\mathcal{Z}}_{W(e_j)} + \frac{1}{2}\nu \cdot (\nabla^{\mathcal{Z}}_{\nu} W)(e_j) \right)\varphi \end{aligned}$$

$$
\begin{aligned}
&\overset{(17)}{=} -\frac{1}{2}\nu\cdot \mathrm{Ric}^{\mathcal{Z}}(\nu)\cdot\varphi+\frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\varphi\\
&\quad +\sum_{j=1}^{n}\varepsilon_j\,\nu\cdot e_j\cdot\Big(\nabla^{\Sigma M}_{W(e_j)}-\frac{1}{2}\nu\cdot W^2(e_j)+\frac{1}{2}\nu\cdot(\nabla^{\mathcal{Z}}_{\nu}W)(e_j)\Big)\varphi\\
&= -\frac{1}{2}\nu\cdot \mathrm{Ric}^{\mathcal{Z}}(\nu)\cdot\varphi+\frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\varphi+\mathfrak{D}^W\varphi\\
&\quad +\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j\,e_j\cdot\Big(-W^2(e_j)+(\nabla^{\mathcal{Z}}_{\nu}W)(e_j)\Big)\varphi.
\end{aligned}
$$

The Riccati equation for the Weingarten map $(\nabla^{\mathcal{Z}}_{\nu}W)(X)=R^{\mathcal{Z}}(X,\nu)\nu+W^2(X)$ yields

$$
\begin{aligned}
i^{-s}[\nabla^{\Sigma\mathcal{Z}}_{\nu},\widetilde{D}]\,\varphi &= -\frac{1}{2}\nu\cdot \mathrm{Ric}^{\mathcal{Z}}(\nu)\cdot\varphi+\frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\varphi+\mathfrak{D}^W\varphi\\
&\quad +\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j\,e_j\cdot(R^{\mathcal{Z}}(e_j,\nu)\nu)\cdot\varphi\\
&= -\frac{1}{2}\nu\cdot \mathrm{Ric}^{\mathcal{Z}}(\nu)\cdot\varphi+\frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\varphi+\mathfrak{D}^W\varphi+\frac{1}{2}\mathrm{ric}^{\mathcal{Z}}(\nu,\nu)\varphi\\
&= \mathfrak{D}^W\varphi-\frac{1}{2}\sum_{j=1}^{n}\varepsilon_i\,\mathrm{ric}^{\mathcal{Z}}(\nu,e_j)\,\nu\cdot e_j\cdot\varphi+\frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\varphi. \qquad (20)
\end{aligned}
$$

The Codazzi-Mainardi equation for $X,Y,V\in TM$ is given by $\big\langle R^{\mathcal{Z}}(X,Y)V,\nu\big\rangle=\big\langle(\nabla^M_X W)(Y),V\big\rangle-\big\langle(\nabla^M_Y W)(X),V\big\rangle$. Thus,

$$
\begin{aligned}
\mathrm{ric}^{\mathcal{Z}}(\nu,X) &= \sum_{j=1}^{n}\varepsilon_j\left\langle R^{\mathcal{Z}}(X,e_j)e_j,\nu\right\rangle\\
&= \sum_{j=1}^{n}\varepsilon_j\left(\left\langle(\nabla^M_X W)(e_j),e_j\right\rangle-\left\langle(\nabla^M_{e_j}W)(X),e_j\right\rangle\right)\\
&= \mathrm{tr}(\nabla^M_X W)-\left\langle\mathrm{div}^M(W),X\right\rangle.
\end{aligned}
$$

Plugging this into (20) we get

$$
\begin{aligned}
i^{-s}[\nabla^{\Sigma\mathcal{Z}}_{\nu},\widetilde{D}]\,\varphi &= \mathfrak{D}^W\varphi-\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j\,\left(\mathrm{tr}(\nabla^M_{e_j}W)-\left\langle\mathrm{div}^M(W),e_j\right\rangle\right)\,\nu\cdot e_j\cdot\varphi\\
&\quad +\frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\varphi.\\
&= \mathfrak{D}^W\varphi-\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j\,e_j(\mathrm{tr}(W))\nu\cdot e_j\cdot\varphi+\frac{1}{2}\nu\cdot\mathrm{div}^M(W)\cdot\varphi\\
&\quad +\frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\varphi.\\
&= \mathfrak{D}^W\varphi-\frac{n}{2}\nu\cdot\mathrm{grad}^M(H)\cdot\varphi+\frac{1}{2}\nu\cdot\mathrm{div}^M(W)\cdot\varphi+\frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\varphi.
\end{aligned}
$$

# 4 The generalized cylinder on semi-Riemannian Spin$^c$ manifolds

Let $M$ be an $n$-dimensional smooth manifold and $g_t$ a smooth 1-parameter family of semi-Riemannian metrics on $M$, $t \in I$ where $I \subset \mathbb{R}$ is an interval. We define the generalized cylinder by

$$\mathcal{Z} := I \times M,$$

with semi-Riemannian metric $g_{\mathcal{Z}} := \langle \cdot, \cdot \rangle = dt^2 + g_t$. The generalized cylinder is an $(n+1)$-dimensional semi-Riemannian manifold of signature $(r+1, s)$ if the signature of $g_t$ is $(r, s)$.

**Proposition 4.1** *There is a 1-1-correspondence between the* $\mathrm{Spin^c}$ *structures on* $M$ *and that on* $\mathcal{Z}$.

**Proof:** As explained in Section 3, $\mathrm{Spin^c}$ structures on $\mathcal{Z}$ can be restricted to $\mathrm{Spin^c}$ structures on $M$. Conversely, given a $\mathrm{Spin^c}$ structure on $M$ it can be pulled back to $I \times M$ via the projection $pr_2 : I \times M \longrightarrow M$ yields a $\mathrm{Spin^c}$ structure on $\mathcal{Z}$. In fact, the pull back of the $\mathrm{Spin}^c(r, s)$-principal bundle $P_{\mathrm{Spin^c}}M$ on $M$ gives rise to a $\mathrm{Spin}^c(r, s)$-principal bundle on $\mathcal{Z}$ denoted by $P_{\mathrm{Spin^c}}\mathcal{Z}$

$$\begin{tikzcd}
P_{\mathrm{Spin^c}}\mathcal{Z} \arrow[r] \arrow[d, "\pi"] & P_{\mathrm{Spin^c}}M \arrow[d, "\pi"'] \\
\mathcal{Z} = I \times M \arrow[r] & M
\end{tikzcd}$$

Enlarging the structure group via the embedding $\mathrm{Spin^c}(r, s) \hookrightarrow \mathrm{Spin^c}(r+1, s)$, which covers the standard embedding

$$\begin{array}{ccc}
\mathrm{SO}(r, s) \times \mathbb{S}^1 & \hookrightarrow & \mathrm{SO}(r+1, s) \times \mathbb{S}^1 \\
(a, z) & \mapsto & \left( \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{a} \end{pmatrix}, z \right),
\end{array}$$

gives a $\mathrm{Spin^c}(r+1, s)$-principal fibre bundle on $\mathcal{Z}$, denoted also by $P_{\mathrm{Spin^c}}\mathcal{Z}$. The pull back of the line bundle $L^M$ on $M$ defining the $\mathrm{Spin^c}$ structure on $M$, gives a line bundle $L^{\mathcal{Z}}$ on $\mathcal{Z}$ such that the following diagram commutes

$$\begin{tikzcd}
L^{\mathcal{Z}} = pr_2^*(L^M) \arrow[r] \arrow[d, "\pi"] & L^M \arrow[d, "\pi"'] \\
\mathcal{Z} = I \times M \arrow[r] & M
\end{tikzcd} \quad .$$

The line bundle $L^{\mathcal{Z}}$ on $\mathcal{Z}$ and the $\mathrm{Spin}^c(r+1, s)$-principal fibre bundle $P_{\mathrm{Spin^c}}\mathcal{Z}$ on $\mathcal{Z}$ yields the $\mathrm{Spin^c}$ structure on $\mathcal{Z}$ which restricts to the given $\mathrm{Spin^c}$ structure on $M$.

**Remark 4.2** *If $M$ is a* $\mathrm{Spin}^c$ *Riemannian manifold and if we denote by $i\Omega^M$ the imaginairy valued curvature on the line bundle $L^M$, we know that there exists a unique curvature 2-form, denoted by $i\Omega^{\mathcal{Z}}$, on the line bundle $L^{\mathcal{Z}} = pr_2^*(L^M)$, defined by $i\Omega^{\mathcal{Z}} = pr_2^*(i\Omega^M)$. Thus we have*

$$\Omega^{\mathcal{Z}}(X,Y) = \Omega^M(X,Y) \text{ and } \Omega^{\mathcal{Z}}(\nu,Y) = 0 \text{ for any } X,Y \in \Gamma(TM).$$

**Proposition 4.3** [2] *On a generalized cylinder $\mathcal{Z} = I \times M$ with semi-Riemannian metric $g^{\mathcal{Z}} = \langle\cdot,\cdot\rangle = dt^2 + g_t$ we define, in every $p \in M$ and $X,Y \in T_pM$, the first and second derivatives of $g_t$ by*

$$\dot{g}_t(X,Y) := \frac{d}{dt}(g_t(X,Y)) \quad \text{and} \quad \ddot{g}_t(X,Y) := \frac{d^2}{dt^2}(g_t(X,Y)).$$

*Hence the following formulas hold:*

$$\begin{aligned}
\langle W(X),Y\rangle &= -\frac{1}{2}\dot{g}_t(X,Y), && (21)\\
\big\langle R^{\mathcal{Z}}(U,V)X,Y\big\rangle &= \big\langle R^{M_t}(U,V)X,Y\big\rangle && (22)\\
&\quad +\frac{1}{4}\Big(\dot{g}_t(U,X)\dot{g}_t(V,Y) - \dot{g}_t(U,Y)\dot{g}_t(V,X)\Big), &&\\
\big\langle R^{\mathcal{Z}}(X,Y)U,\nu\big\rangle &= \frac{1}{2}\Big((\nabla^{M_t}_Y\dot{g}_t)(X,U) - (\nabla^{M_t}_X\dot{g}_t)(Y,U)\Big), && (23)\\
\big\langle R^{\mathcal{Z}}(X,\nu)\nu,Y\big\rangle &= -\frac{1}{2}\Big(\ddot{g}_t(X,Y) + \dot{g}_t(W(X),Y)\Big), && (24)
\end{aligned}$$

*where $X,Y,U,V \in T_pM$, $p \in M$.*

# 5 The variation formula for the Dirac operator on Spin$^c$ manifolds

First we give some facts about parallel transport on $\mathrm{Spin}^c$ manifolds along a curve $c$. We consider a Riemannian $\mathrm{Spin}^c$ manifold $N$, we know that there exists a unique correspondence which associates to a spinor field $\psi(t) = \psi(c(t))$ along a curve $c : I \longrightarrow N$ another spinor field $\frac{D}{dt}\psi$ along $c$, called the covariant derivative of $\psi$ along $c$, such that

$$\frac{D}{dt}(\psi+\varphi) = \frac{D}{dt}\psi + \frac{D}{dt}\varphi, \text{ for any } \psi \text{ and } \varphi \text{ along the curve } c,$$

$$\frac{D}{dt}(f\psi) = f\frac{D}{dt}\psi + (\frac{d}{dt}f)\,\psi, \text{ where } f \text{ is a differentiable function on } I,$$

$$\nabla^{\Sigma N}_{\dot{c}(t)}\psi = \frac{D}{dt}\varphi, \text{ where } \varphi(t) = \psi(c(t)).$$

A spinor field $\psi$ along a curve $c$ is called parallel when $\frac{D}{dt}\psi(t) = 0$ for all $t \in I$. Now, if $\psi_0$ is a spinor at the point $c(t_0)$, $t_0 \in I, (\psi_0 \in \Sigma_{c(t_0)}N)$ then there exists a unique parallel spinor $\varphi$ along $c$, such that $\psi_0 = \varphi(t_0)$. The linear isometry $\tau_{t_0}^{t_1}$ defined by

$$\begin{aligned} \tau_{t_0}^{t_1} : \Sigma_{c(t_0)}N &\longrightarrow \Sigma_{c(t_1)}N \\ \psi_0 &\longrightarrow \varphi(t_1), \end{aligned}$$

is called the parallel transport along the curve $c$ from $c(t_0)$ to $c(t_1)$. The basic property of the parallel transport on a $\mathrm{Spin^c}$ manifold is the following: Let $\psi$ be a spinor field on a Riemannian $\mathrm{Spin^c}$ manifold $N$ , $X \in \Gamma(TN)$, $p \in N$ and $c : I \longrightarrow N$ an integral curve through $p$, i.e., $c(t_0) = p$ and $\frac{d}{dt}c(t) = X(c(t))$, we have

$$(\nabla_X^{\Sigma N}\psi)_p = \frac{d}{dt}\Big(\tau_t^{t_0}(\psi(t))\Big)|_{t=t_0}. \tag{25}$$

Now, we consider $g_t$ a smooth 1-parameter family of semi-Riemannian metrics on a $\mathrm{Spin^c}$ manifold $M$ and the generalized cylinder $\mathcal{Z} = I \times M$ with semi-Riemannian metric $g^{\mathcal{Z}} = \langle\cdot,\cdot\rangle = dt^2 + g_t$. For $t \in I$ we denote by $M_t$ the manifold $(M, g_t)$. Let us write "$\cdot$" for the Clifford multiplication on $\mathcal{Z}$ and "$\bullet_t$" for that on $M_t$. Recall from Section 4 that $\mathrm{Spin^c}$ structures on $M$ and $\mathcal{Z}$ are in 1-1-correspondence and $\Sigma\mathcal{Z}|_{M_t} = \Sigma M_t$ as hermitian vector bundles if $n = r + s$ is even and $\Sigma^+\mathcal{Z}|_{M_t} = \Sigma M_t$ if $n$ is odd. For a given $x \in M$ and $t_0, t_1 \in I$, parallel transport $\tau_{t_0}^{t_1}$ on the generalized cylinder $\mathcal{Z}$ along the curve $c : I \to I \times M, t \to (t, x)$ is given by

$$\tau_{t_0}^{t_1} : \Sigma_{c(t_0)}\mathcal{Z} \simeq \Sigma_x M_{t_0} \longrightarrow \Sigma_{c(t_1)}\mathcal{Z} \simeq \Sigma_x M_{t_1}.$$

This isomorphism satisfies

$$\tau_{t_0}^{t_1}(X \bullet_{t_0} \varphi) = (\zeta_{t_0}^{t_1}X) \bullet_{t_1} (\tau_{t_0}^{t_1}\varphi),$$

$$< \tau_{t_0}^{t_1}\psi, \tau_{t_0}^{t_1}\varphi > = < \psi, \varphi >,$$

where $\zeta_{t_0}^{t_1} : T_{(x,t_0)}\mathcal{Z} \simeq T_x M_{t_0} \to T_{(x,t_1)}\mathcal{Z} \simeq T_x M_{t_1}$ is the parallel transport on $\mathcal{Z}$ along the same curve $c$, $X \in T_x M_{t_0}$ and $\psi, \varphi \in \Sigma_x M_{t_0}$.

**Theorem 5.1** *On a* $\mathrm{Spin^c}$ *manifold $M$, let $g_t$ be a smooth 1-parameter family of semi-Riemannian metrics. Denote by $D^{M_t}$ the Dirac operator of $M_t$, and $\mathfrak{D}^{\dot g_t} = \sum_{i,j=1}^n \varepsilon_i\varepsilon_j \dot g_t(e_i, e_j)e_i \bullet_t \nabla_{e_j}^{\Sigma M_t}$. Then for any smooth spinor field $\psi$ on $M_{t_0}$ we have*

$$\frac{d}{dt}\bigg|_{t=t_0} \tau_t^{t_0} D^{M_t}\tau_{t_0}^t\psi = -\frac{1}{2}\mathfrak{D}^{\dot g_{t_0}}\psi + \frac{1}{4}\,\mathrm{grad}^{M_{t_0}}(\mathrm{tr}_{g_{t_0}}(\dot g_{t_0})) \bullet_{t_0} \psi - \frac{1}{4}\,\mathrm{div}^{M_{t_0}}(\dot g_{t_0}) \bullet_{t_0} \psi.$$

**Proof:** The vector field $\nu := \frac{\partial}{\partial t}$ is spacelike of unit length and orthogonal to the hypersurfaces $M_t := \{t\} \times M$. Denote by $W_t$ the Weingarten map of $M_t$ with respect to

$\nu$ and by $H_t$ the mean curvature. If $X$ is a local coordinate field on $M$, then $\langle X, \nu\rangle = 0$ and $[X, \nu] = 0$. Thus

$$\begin{aligned} 0 &= d_\nu \langle X, \nu\rangle = \left\langle \nabla^{\mathcal{Z}}_\nu X, \nu\right\rangle + \left\langle X, \nabla^{\mathcal{Z}}_\nu \nu\right\rangle = \left\langle \nabla^{\mathcal{Z}}_X \nu, \nu\right\rangle + \left\langle X, \nabla^{\mathcal{Z}}_\nu \nu\right\rangle \\ &= -\langle W_t(X), \nu\rangle + \left\langle X, \nabla^{\mathcal{Z}}_\nu \nu\right\rangle = \left\langle X, \nabla^{\mathcal{Z}}_\nu \nu\right\rangle \end{aligned}$$

and differentiating $\langle \nu, \nu\rangle = 1$ yields $\left\langle \nu, \nabla^{\mathcal{Z}}_\nu \nu\right\rangle = 0$. Hence $\nabla^{\mathcal{Z}}_\nu \nu = 0$, i.e., for $x \in M$ the curves $t \mapsto (t, x)$ are geodesics parametrized by arclength. So the assumptions of Proposition 3.4 are satisfied for the foliation $(M_t)_{t\in I}$. By Remark 4.2, the commutator formula of Proposition 3.4 gives for a section $\varphi$ of $\Sigma M_t$, (or $\Sigma^+ M_t$ if $n$ is odd)

$$i^{-s}[\nabla^{\Sigma\mathcal{Z}}_\nu, D^{M_t}]\,\varphi = \mathfrak{D}^{W_t}\varphi - \frac{n}{2}\,\mathrm{grad}^{M_t}(H_t) \bullet_t \varphi \;\; + \;\; \frac{1}{2}\,\mathrm{div}^{M_t}(W_t) \bullet_t \varphi. \quad (26)$$

From Proposition 4.3 we deduce

$$\mathrm{div}^{M_t}(W_t) = -\frac{1}{2}\mathrm{div}^{M_t}(\dot{g}_t), \qquad H_t = -\frac{1}{2n}\mathrm{tr}_{g_t}(\dot{g}_t) \quad \text{and} \quad \mathfrak{D}^{W_t} = -\frac{1}{2}\mathfrak{D}^{\dot{g}_t}.$$

Thus (26) can be rewritten as

$$i^{-s}[\nabla^{\Sigma\mathcal{Z}}_\nu, D^{M_t}]\,\varphi = -\frac{1}{2}\mathfrak{D}^{\dot{g}_t}\varphi + \frac{1}{4}\,\mathrm{grad}^{M_t}(\mathrm{tr}_{g_t}(\dot{g}_t)) \bullet_t \varphi - \frac{1}{4}\,\mathrm{div}^{M_t}(\dot{g}_t) \bullet_t \varphi. \quad (27)$$

Now if $\varphi$ is parallel along the curves $t \mapsto (t, x)$, i.e., it is of the form $\varphi(t, x) = \tau^t_{t_0}\psi(t_0, x)$ for some spinor field $\psi$ on $M_{t_0}$, then using (25) at $t = t_0$, the left hand side of (27) could be written as

$$\begin{aligned} i^{-s}[\nabla^{\Sigma\mathcal{Z}}_\nu, D^{M_t}]\,\varphi &= i^{-s}\nabla^{\Sigma\mathcal{Z}}_\nu D^{M_t}\,\varphi = i^{-s}\left.\frac{d}{dt}\right|_{t=t_0} \tau^{t_0}_t D^{M_t}\,\varphi \\ &= i^{-s}\left.\frac{d}{dt}\right|_{t=t_0} \tau^{t_0}_t D^{M_t}\tau^t_{t_0}\psi, \end{aligned} \quad (28)$$

which gives the variation formula for the Dirac operator.

**Corollary 5.2** *Let $(M^n, g)$ be a $\mathrm{Spin}^c$ Riemannian manifold, if we consider the family of metrics defined by $g_t = g + tk$, where $k$ is a symmetric $(0, 2)$-tensor, we have*

$$\left.\frac{d}{dt}\right|_{t=0} \tau^0_t D^{M_t}\tau^t_0\psi = -\frac{1}{2}\mathfrak{D}^k\psi + \frac{1}{4}\,\mathrm{grad}^M(\mathrm{tr}_g(k)) \cdot \psi - \frac{1}{4}\,\mathrm{div}^M(k) \cdot \psi, \quad (29)$$

*where "$\cdot = \bullet_{t_0=0}$" is the Clifford multiplication on $M$.*

This formula has been proved in [4], Theorem 21 for spin Riemannian manifolds and in [2] for spin semi-Riemannian manifolds.

# 6 Energy-Momentum tensor on Spin$^c$ manifolds

In this section we study the Energy-Momentum tensor on $\mathrm{Spin^c}$ Riemannian manifolds from a geometric point of vue. We begin by giving the proofs of Proposition 1.1, Theorem 1.2 and Proposition 1.3.

**Proof of Proposition 1.1 :** Using Equation (29) we calculate

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} (\tau_t^0 D^{M_t}\tau_0^t\psi,\psi)_{g_t} &= \frac{d}{dt}\bigg|_{t=0} (D^{M_t}\tau_0^t\psi,\tau_0^t\psi)_{g_t} \\
&= -\frac{1}{2}(\mathfrak{D}^k\psi,\psi)_g \\
&= -\frac{1}{2}\sum_{i,j} k(e_i,e_j)(e_i\cdot\nabla^{\Sigma M}_{e_j}\psi,\psi) \\
&= -\frac{1}{2}\int_M < k, T_\psi > dv_g.
\end{aligned}$$

**Proof of Theorem 1.2** : The Proof of this Theorem will be omitted since it is similar to the one given by Friedrich and Kim in [8] for spin manifolds.

**Proof of Proposition 1.3** : Let $\psi$ be any parallel spinor field on $\mathcal{Z}$. Then Equation (17) yields

$$\nabla^{\Sigma M}_X\varphi = \frac{1}{2}W(X)\bullet\varphi. \tag{30}$$

Let $(e_1, ..., e_n)$ be a positively oriented local orthonormal basis of $TM$. For $j = 1, ..., n$ we have

$$\nabla^{\Sigma M}_{e_j}\varphi = \frac{1}{2}\sum_{k=1}^n W_{jk}\ e_k\bullet\varphi.$$

Taking Clifford multiplication by $e_i$ and the scalar product with $\varphi$, we get

$$\mathrm{Re}\left\langle e_i\bullet\nabla^{\Sigma M}_{e_j}\varphi,\varphi\right\rangle = \frac{1}{2}\sum_{k=1}^n W_{jk}\mathrm{Re}\left\langle e_i\bullet e_k\bullet\varphi,\varphi\right\rangle.$$

Since $\mathrm{Re}\left\langle e_i\bullet e_k\bullet\varphi,\varphi\right\rangle = -\delta_{ik}|\varphi|^2$, it follows, by the symmetry of $W$

$$\mathrm{Re}\left\langle e_i\bullet\nabla^{\Sigma M}_{e_j}\varphi + e_j\bullet\nabla^{\Sigma M}_{e_i}\varphi,\varphi\right\rangle = -W_{ij}|\varphi|^2.$$

Therefore, $2\ell^\varphi = -W$. Using Equation (18) it is easy to see that $\varphi$ is an eigenspinor associated with the eigenvalue $-\frac{n}{2}H$ of $\widetilde{D}$. Since $\mathrm{Scal}^{\mathcal{Z}} = \mathrm{Scal}^M + 2\ \mathrm{ric}^{\mathcal{Z}}(\nu,\nu) - n^2H^2 + |W|^2$ we get

$$\frac{1}{4}(\mathrm{Scal}^M - c_n|\Omega^M|) + |T^\varphi|^2 = \frac{1}{4}(\mathrm{Scal}^{\mathcal{Z}} - 2\ \mathrm{ric}^{\mathcal{Z}}(\nu,\nu) - c_n|\Omega^M|) + n^2\frac{H^2}{4},$$

hence $M$ satisfies the equality case in (3) if and only if (6) holds.

**Corollary 6.1** *Under the same conditions as Proposition 1.3, if* $n = 2$ *or* $3$*, the hypersurface* $M$ *satisfies the equality case in (3) if* $\mathrm{Ric}^{\mathcal{Z}}(\nu) = 0$ *and* $\mathrm{Scal}^{\mathcal{Z}} \geqslant 0$*.*

**Proof:** Since $\mathcal{Z}$ has a parallel spinor, we have (see [7])

$$|\mathrm{Ric}^{\mathcal{Z}}(\nu)| = |\nu \lrcorner \Omega^{\mathcal{Z}}|, \tag{31}$$

$$i(Y \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi = \mathrm{Ric}^{\mathcal{Z}}(Y) \cdot \psi \ \text{ for every } \ Y \in \Gamma(\Sigma\mathcal{Z}). \tag{32}$$

For $Y = e_j$ in Equation (32) then taking Clifford multiplication by $e_j$ and summing from $j = 1, ..., n+1$, we get

$$i \sum_{j=1}^{n+1} e_j \cdot (e_j \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi = \sum_{j=1}^{n+1} e_j \cdot \mathrm{Ric}^{\mathcal{Z}}(e_j) \cdot \psi = -\mathrm{Scal}^{\mathcal{Z}} \psi.$$

But $2\ \Omega^{\mathcal{Z}} \cdot \psi = \sum_{j=1}^{n+1} e_j \cdot (e_j \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi$, hence we deduce that $\Omega^{\mathcal{Z}} \cdot \psi = i \frac{\mathrm{Scal}^{\mathcal{Z}}}{2} \psi$. By (31) and (15) we obtain $\Omega^M \bullet \varphi = i \frac{\mathrm{Scal}^{\mathcal{Z}}}{2} \varphi$. Since $n = 2$ or $3$ we have $|\Omega^M| = \frac{\mathrm{Scal}^{\mathcal{Z}}}{2}$ and Equation (6) is satisfied.

**Corollary 6.2** *Under the same conditions as Proposition 1.3, if the restriction of the complex line bundle* $L^{\mathcal{Z}}$ *is flat, i.e.,* $L^M$ *is a flat complex line bundle (*$\Omega^M = 0$*), the hypersurface* $M$ *is a limiting manifold for (3).*

**Proof:** Since $\Omega^M = 0$, Equation (15) yields $i \frac{\mathrm{Scal}^{\mathcal{Z}}}{2} \varphi = \Omega^{\mathcal{Z}} \cdot \psi_{|_M} = (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi$. But,

$$\begin{aligned} i(\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi &= i(\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi)_{|_M} = (\nu \cdot \mathrm{Ric}^{\mathcal{Z}}(\nu) \cdot \psi)_{|_M} \\ &= -\mathrm{ric}^{\mathcal{Z}}(\nu, \nu)\varphi + \sum_{j=1}^{n} \mathrm{ric}^{\mathcal{Z}}(\nu, e_j)\, e_j \bullet \varphi. \end{aligned} \tag{33}$$

Taking the real part of the scalar product of Equation (33) with $\varphi$ yields $\frac{\mathrm{Scal}^{\mathcal{Z}}}{2} = \mathrm{ric}^{\mathcal{Z}}(\nu, \nu)$, hence Equation (6) is satisfied.

Now, let $M$ be a $\mathrm{Spin}^{\mathrm{c}}$ Riemannian manifold having a generalized Killing spinor field $\varphi$ with a symmetric endomorphism $F$ on the tangent bundle $TM$. As mentioned in the introduction, it is straightforward to see that $2T^{\varphi}(X, Y) = -\langle F(X), Y \rangle$. We will study these generalized Killing spinors when the tensor $F$ is a Codazzi-Mainardi tensor, i.e., $F$ satisfies

$$(\nabla^M_X F)(Y) = (\nabla^M_Y F)(X) \ \text{ for } \ X, Y \in \Gamma(TM). \tag{34}$$

For this, we give the following lemma whose proof will be omitted since it is similar to Lemma 7.3 in [2].

**Lemma 6.3** [2] *Let $g_t$ be a smooth 1-parameter family of semi-Riemannian metrics on a* $\mathrm{Spin^c}$ *manifold $(M^n, g = g_0)$ and let $F$ be a field of symmetric endomorphisms of $TM$. We consider the metric $g_{\mathcal{Z}} = \langle .,. \rangle = dt^2 + g_t$ on $\mathcal{Z}$ such that $g_t(X,Y) = g((\mathrm{Id} - tF)^2 X, Y)$ for all vector fields $X, Y$ on $M$. We have $\left\langle R^{\mathcal{Z}}(U,\nu)\nu, V \right\rangle = 0$ for all vector fields $U, V$ tangent to $M$ and if $F$ satisfies the Codazzi-Mainardi equation then $\left\langle R^{\mathcal{Z}}(U,V)W, \nu \right\rangle = 0$ for all $U, V$ and $W$ on $\mathcal{Z}$.*

**Proof of Theorem 1.4**: We define $\psi_{(0,x)} := \varphi_x$ via the identification $\Sigma_x M \cong \Sigma_{(0,x)}\mathcal{Z}$ (resp. $\Sigma^+_{(0,x)}\mathcal{Z}$ for $n$ odd) and $\psi_{(t,x)} = \tau_0^t \psi_{(0,x)}$. By Equation (21), the endomorphism $F$ is the Weingarten tensor of the immersion of $\{0\} \times M$ in $\mathcal{Z}$ and hence by construction we have for all $X \in \Gamma(TM)$

$$\nabla^{\Sigma\mathcal{Z}}_X \psi|_{\{0\}\times M} = 0 \quad \text{and} \quad \nabla^{\Sigma\mathcal{Z}}_\nu \psi \equiv 0. \tag{35}$$

Since the tensor $F$ satisfies the Codazzi-Mainardi equation, Lemma 6.3 yields $g_{\mathcal{Z}}(R^{\mathcal{Z}}(U,V)W,\nu) = 0$ for all $U, V$ and $W \in \Gamma(\mathcal{Z})$ and $g_{\mathcal{Z}}(R^{\mathcal{Z}}(X,\nu)\nu, Y) = 0$ for all $X$ and $Y$ tangent to $M$. Hence $R^{\mathcal{Z}}(\nu, X) = 0$ for all $X \in \Gamma(TM)$. Let $X$ be a fixed arbitrary tangent vector field on $M$. Using (10) and (35) we get

$$\nabla^{\Sigma\mathcal{Z}}_\nu \nabla^{\Sigma\mathcal{Z}}_X \psi = \mathcal{R}^{\Sigma\mathcal{Z}}(\nu, X)\psi = \frac{1}{2} R^{\mathcal{Z}}(X,\nu)\cdot\psi + \frac{i}{2}\Omega^{\mathcal{Z}}(X,\nu)\psi = 0.$$

Thus showing that the spinor field $\nabla^{\Sigma\mathcal{Z}}_X \psi$ is parallel along the geodesics $\mathbb{R} \times \{x\}$. Now (35) shows that this spinor vanishes for $t = 0$, hence it is zero everywhere on $\mathcal{Z}$. Since $X$ is arbitrary, this shows that $\psi$ is parallel on $\mathcal{Z}$.

**Corollary 6.4** *Let $(M^3, g)$ be a compact, oriented Riemannian manifold and $\varphi$ an eigenspinor associated with the first eigenvalue $\lambda_1$ of the Dirac operator such that the Energy-Momentum tensor associated with $\varphi$ is a Codazzi tensor. $M$ is a limiting manifold for (3) if and only if the generalized cylinder $\mathcal{Z}^4$, equipped with the* $\mathrm{Spin^c}$ *structure arising from the given one on $M$, is Kähler of positive scalar curvature and the immersion of $M$ in $\mathcal{Z}$ has constant mean curvature $H$.*

**Proof:** First, we should point out that every 3-dimensional compact, oriented, smooth manifold has a $\mathrm{Spin^c}$ structure. Now, if $M^3$ is a limiting manifold for (3), by Theorem 1.4, the generalized cylinder has a parallel spinor whose restriction to $M$ is $\varphi$. Since $\mathcal{Z}$ is a 4-dimensional $\mathrm{Spin^c}$ manifold having parallel spinor, $\mathcal{Z}$ is Kähler [1]. Moreover, using (15), we have

$$\Omega^M \bullet \varphi = i\frac{\mathrm{Scal}^{\mathcal{Z}}}{2}\varphi = i\frac{c_n}{2}|\Omega^M|\varphi,$$

so $\mathrm{Scal}^{\mathcal{Z}} \geqslant 0$. Finaly $H = \frac{1}{n}\mathrm{tr}(W) = \frac{1}{n}\mathrm{tr}(-2T^\varphi) = -\frac{2}{n}\lambda_1$, which is a constant. Now if the generalized cylinder is Kähler and $M$ is a compact hypersurface of constant mean curvature $H$, thus $M$ is compact hypersurface immersed in a $\mathrm{Spin^c}$ manifold having parallel spinor with constant mean curvature. Since $\mathrm{Scal}^{\mathcal{Z}} \geqslant 0$ and $\nu \lrcorner \Omega^{\mathcal{Z}} = \mathrm{Ric}^{\mathcal{Z}}(\nu) = 0$, Corollary 6.1 gives the result.

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